Optical BCS conductivity at imaginary frequencies and dispersion energies of superconductors

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Abstract
We present an efficient expression for the analytic continuation to arbitrary complex frequencies of the complex optical and ac conductivity of a homogeneous superconductor with an arbitrary mean free path. Knowledge of this quantity is fundamental in the calculation of thermodynamic potentials and dispersion energies involving type-I superconducting bodies. When considered for imaginary frequencies, our formula evaluates faster than previous schemes involving Kramers–Kronig transforms. A number of applications illustrate its efficiency: a simplified low-frequency expansion of the conductivity, the electromagnetic bulk self-energy due to longitudinal plasma oscillations, and the Casimir free energy of a superconducting cavity.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The linear response of a homogeneous system to an external field can be conveniently described by a susceptibility $\chi(\mathbf{q}, \omega)$ in the Fourier space. Calculations at real frequencies can be directly compared to experimental spectra. Imaginary frequencies, $\omega = \mathrm{i}\xi$, are theoretically useful to deal with equilibrium thermodynamical properties [1]. In particular, quantum field theories, after a Wick rotation in imaginary time, can be studied at nonzero temperature by expressing partition functions as Matsubara sums over discrete frequencies $\omega_n = 2\pi n \mathrm{i} T$ (bosons: $n = 0, 1, \ldots$; fermions: $n = \frac{1}{2}, \frac{3}{2}, \ldots$) [2]. A further advantage of this approach comes from the fact that causal response functions are well behaved on the (positive) imaginary axis, a feature that facilitates both analytical and numerical calculations [1].

A strong motivation for the present paper is the interest in electromagnetic dispersion forces for superconducting materials. In the context of the electromagnetic Casimir effect, the role of absorption in a real mirror is still poorly understood. The superconducting phase transition provides a clean way to change absorption over a narrow temperature range, as has been suggested in [3]. The standard theory of Bardeen–Cooper–Schrieffer (BCS) [4] is well established for conventional superconductors and has provided the basis for calculating the optical conductivity $\sigma(\omega)$. In their seminal paper [5], Mattis and Bardeen included the impact of disorder in the 'dirty limit' where the scattering mean free path $\ell = v_F \tau$ is small. This approach has been generalized to arbitrary purity [6–8].

The conductivity, at imaginary frequencies, is the basic response function that yields electromagnetic dispersion forces between superconducting bodies at nonzero temperature [1, 9, 10]. In the previous work, the common strategy was to compute $\sigma(\mathrm{i}\xi)$ numerically, starting from the conductivity available at real frequencies and performing a Kramers–Kronig transform [11]. This technique is quite inefficient because in the BCS theory, $\sigma(\omega)$ is itself an integral over the quasiparticle spectrum. This fact also complicates analytical studies. We present in this paper an alternative formula for the BCS conductivity that does not need a Kramers–Kronig transform, is given by a rapidly converging integral and can be evaluated in the upper half of the complex frequency plane. Our derivation is based on a consistent regularization scheme where the quasiparticle response, far away from the Fermi edge (that coincides with a normal conductor), is subtracted.

The paper is organized as follows: in section 2 we review the BCS conductivity at real frequencies. In section 3 we present our formula for the BCS conductivity $\sigma(z)$ at arbitrary complex frequencies z, and we show that for $z = \mathrm{i}\xi$, it coincides with the Kramers–Kronig transform, evaluated numerically from $\sigma(\omega)$ given in [7]. In section 4 we discuss several applications. First, we derive the low-frequency asymptotics and recover in a simple way the findings of [8]. We then compare this with the two-fluid model of a superconductor [4] and show its limitations. As further applications, we study the zero temperature electromagnetic self-energy of a bulk superconductor, as a function of the scattering mean free path (section 4.4), and present a numerical calculation of the Casimir energy between two superconducting plates near the critical temperature (section 4.5).

Our conclusions are presented in section 5. Two appendices close the paper: in appendix A we present an alternative computation of the normal-metal conductivity, more in the spirit of [5]. Finally, in appendix B, we verify analytically that our formula, when considered for real frequencies, reproduces the results of [7].

2. Mattis–Bardeen theory

In 1958 Bardeen and Mattis developed a calculation of the anomalous skin effect in superconductors [5]. The conductivity of a bulk of superconducting material can be obtained from the linear current response to an applied vector potential. Expanding the current over the quasiparticle spectrum given by the BCS theory, one finds a Chambers-like expression for the current density [4, 5]:

$$j(\mathbf{r}, \omega) = \frac{3\sigma_0}{(2\pi)^2 \ell} \int \mathrm{d}^3 R \, \frac{\mathbf{R}(\mathbf{R} \cdot \mathbf{A}(\mathbf{r} + \mathbf{R}))}{R^4} \, \mathrm{e}^{-R/\ell} I(\omega, R) \tag{1}$$

$$I(\omega, R) = \int \mathrm{d}\epsilon \, \mathrm{d}\epsilon' \left\{ L(\omega, \epsilon, \epsilon') - \frac{f(\epsilon) - f(\epsilon')}{\epsilon' - \epsilon} \right\} \cos[R(\epsilon' - \epsilon)/v_F], \tag{2}$$

where σ_0 is the dc conductivity in the normal state. Disorder is taken into account via the scattering mean free path ℓ, v_F is the Fermi velocity, $f(\epsilon)$ is the Fermi–Dirac function and ϵ is measured from the Fermi edge.

The spectral function $L(\omega, \epsilon, \epsilon')$ involves the quasiparticle energies ($\hbar = 1$):

$$E = \sqrt{\epsilon^2 + \Delta^2} \qquad [\Delta \equiv \Delta(T) \text{: BCS gap}] \tag{3}$$

and is given by

$$L(\omega, \epsilon, \epsilon') = \frac{1}{2} p(\epsilon, \epsilon') \left(\frac{f(E') - f(E)}{E - E' - \omega - i0^+} + \frac{f(E') - f(E)}{E - E' + \omega + i0^+} \right)$$
$$+ \frac{1}{2} q(\epsilon, \epsilon') \left(\frac{1 - f(E') - f(E)}{E + E' - \omega - i0^+} + \frac{1 - f(E') - f(E)}{E + E' + \omega + i0^+} \right) \tag{4}$$

with the coherence factors

$$\left. \begin{array}{r} p(\epsilon, \epsilon') \\ q(\epsilon, \epsilon') \end{array} \right\} = \frac{EE' \pm (\epsilon\epsilon' + \Delta^2)}{2EE'}. \tag{5}$$

The infinitesimal parameter 0^+, set to zero at the end of the calculation, ensures an adiabatic switching-on of the external field with time-dependence $e^{-i\omega t}$ and a causal response. The term $(f - f')/(\epsilon' - \epsilon)$ that appears in equation (2) gives an imaginary contribution to the conductivity (for real ω) and represents the diamagnetic (London) current [5, 12]. Note that the same expression is found by the Green's function approach of Belitz *et al* [6] when the spectral strength of the material in its normally conducting state is considered as diffusive within a charge-conserving approximation.

In this paper, we focus on the local limit of the current response: it is obtained from the previous expression assuming that the vector potential $\mathbf{A}(\mathbf{r} + \mathbf{R})$ is slowly varying in space. Taking it out of the integral and integrating over $\mathbf{R}$, one obtains the conductivity in the form [8]

$$\sigma(\omega) = \frac{\sigma_0 \gamma}{i\omega} \int d\epsilon \, d\epsilon' \left\{ L(\omega, \epsilon, \epsilon') - \frac{f - f'}{\epsilon' - \epsilon} \right\} D_\gamma(\epsilon - \epsilon') \tag{6}$$

$$D_\gamma(x) := \frac{\gamma/\pi}{x^2 + \gamma^2}, \tag{7}$$

where the scattering rate $\gamma = 1/\tau = v_F/\ell$ can be interpreted as the effective width of quasiparticle energies at the Fermi edge (last factor in equation (6)). This expression has been further simplified by Zimmermann [7] and Berlinsky *et al* [8], but their final result was given in a form which involves the frequency ω in the integration boundaries and in the argument of Heaviside functions, so that a direct continuation to complex frequencies is not straightforward. In the next section, we show that the analytic continuation of the BCS conductivity (6) to arbitrary complex frequencies z in the upper half-plane C^+ can be written in the form

$$\sigma(z) = \frac{\sigma_0}{1 - iz\tau} + \delta\sigma_{\text{BCS}}(z), \tag{8}$$

where one recognizes in the first term the analytical continuation of the Drude result for the normal metal $\sigma_{\text{Dr}}(\omega) = \sigma_0/(1 - i\omega\tau)$. The BCS correction $\delta\sigma_{\text{BCS}}(z)$ is given by equations (19), (21) below that can be easily evaluated numerically.

3. Analytic continuation to complex frequencies

Let us now go back to the expression given in equation (6). We note that the spectral function (4) is evidently analytic for $\omega = z \in \mathbb{C}^+$, and the analytic continuation of the BCS conductivity (6) is immediate at this stage. The double integral over ϵ and ϵ' is, however, tricky to perform.

Mattis and Bardeen rewrite the integrand using symmetries under the exchange of $\epsilon \leftrightarrow \epsilon'$ and introduce a regulating function to ensure the convergence at large ϵ, ϵ'. This makes obsolete the inclusion of the diamagnetic term under the integral in equation (6). They then perform the integration along a contour in the complex ϵ'-plane, leaving only the integration over ϵ to be done numerically. Further calculations are performed in the 'dirty' limit where the mean free path ℓ is much smaller than the coherence length v_F/Δ, i.e. $\gamma \gg \Delta$. Zimmermann [7] and Berlinsky *et al* [8] consider a general scattering rate, but focus on the case of real frequencies.

In the following, we show that the difficulty with the convergence can be overcome by a proper handling of the integration scheme: we exploit the exchange symmetry in the $\epsilon\epsilon'$-plane and secure uniform convergence of the double integral by a suitable subtraction. We take into account explicitly the diamagnetic current and include a general scattering rate as in [7, 8]. This procedure makes the Drude term in equation (8) emerge in a natural way for complex frequencies and leads to a numerically convenient form for the remaining BCS correction (whose integrand is strongly localized).

3.1. Normal metal

Let us consider first the limiting case of a normal conductor, that is the expression obtained from equation (6) in the limit $\Delta \to 0$. We allow for complex ω, denoted by z. The coherence factors in equation (5) reduce to $\frac{1}{2}[1 \pm \mathrm{sgn}(\epsilon\epsilon')]$. We observe that the first and second terms of equation (4) are related by the mapping $E' \leftrightarrow -E'$. It is easy to check that in all four quadrants of the $\epsilon\epsilon'$-plane, we can replace the spectral function by

$$\Delta \to 0 : L(z, \epsilon, \epsilon') \mapsto \frac{\epsilon' - \epsilon}{(\epsilon' - \epsilon)^2 - z^2}[f(\epsilon) - f(\epsilon')]. \tag{9}$$

The conductivity of the normal metal is now obtained as follows. Introducing the variables $s = \frac{1}{2}(\epsilon + \epsilon')$ and $\delta = \epsilon' - \epsilon$, the difference of Fermi functions can be written as [$k_B = 1$]

$$f(\epsilon) - f(\epsilon') = \frac{\sinh(\delta/2T)}{\cosh(\delta/2T) + \cosh(s/T)} \tag{10}$$

which is integrable at $|s| \to \infty$ with the result

$$\int_{-\infty}^{+\infty} \mathrm{d}s[f(\epsilon) - f(\epsilon')] = \delta. \tag{11}$$

The integration over δ is then easily performed:

$$\sigma(z) = \frac{\sigma_0 \gamma}{\mathrm{i}z} \int_{-\infty}^{+\infty} \mathrm{d}\delta \left\{ \frac{\delta^2}{\delta^2 - z^2} - 1 \right\} D_\gamma(\delta) = \frac{\sigma_0 \gamma}{\gamma - \mathrm{i}z}, \tag{12}$$

where the subtraction in curly brackets corresponds to the limit $z \to 0$ (the diamagnetic current, compare equations (6) and (9)). We thus find the analytic continuation of the Drude conductivity. Appendix A presents an alternative derivation of this result, that proceeds in close analogy to [5]. In this approach, the integrand is multiplied by a factor $\exp[\mathrm{i}(\epsilon + \epsilon')/(2\Lambda)]$ in order to enforce absolute convergence in the direction $|\epsilon + \epsilon'| \to \infty$ and the limit $\Lambda \to \infty$ is taken in the end.

3.2. Superconductor

We now return to the full BCS theory. It is a simple matter to verify that for any complex frequency $z \in \mathbb{C}^+$, the integrand involving the spectral function $L(z, \epsilon, \epsilon')$ can be split into two terms:

$$L(z, \epsilon, \epsilon') D_\gamma(\epsilon - \epsilon') = F(z, \epsilon, \epsilon') + F(z, \epsilon', \epsilon), \tag{13}$$

where

$$F(z, \epsilon, \epsilon') = -D_\gamma(\epsilon - \epsilon') \frac{\tanh(E/2T)}{4E} \left\{ \frac{A_+ + \epsilon\epsilon'}{\epsilon'^2 - Q_+^2} + \frac{A_- + \epsilon\epsilon'}{\epsilon'^2 - Q_-^2} \right\}, \tag{14}$$

$$A_\pm(z, E) = E(E \pm z) + \Delta^2, \tag{15}$$

$$Q_\pm^2(z, E) = (E \pm z)^2 - \Delta^2. \tag{16}$$

Equation (13) shows manifestly the symmetry of the integrand under the exchange $\epsilon \leftrightarrow \epsilon'$. Mattis and Bardeen proceed by integrating separately over the two terms $F(z, \epsilon, \epsilon')$ and $F(z, \epsilon', \epsilon)$, but to do so, one must ensure absolute convergence of the double integral. Indeed, for large $|\epsilon|$, $|\epsilon'|$, the tanh functions tend to ± 1, and from equations (14)–(16), we see that $F(z, \epsilon, \epsilon')$ becomes proportional to $(\epsilon - \epsilon')^{-1}$. Therefore, for large $|\epsilon|$, $|\epsilon'|$, the integrand $F(z, \epsilon, \epsilon')$ becomes independent of the variable $s = (\epsilon + \epsilon')/2$, and then its double integral is not absolutely convergent as $|s| \to \infty$.

This divergence can be conveniently cured by the following subtraction procedure. We evaluate the asymptotic limit of $F(z, \epsilon, \epsilon')$ as $|\epsilon + \epsilon'| \gg \Delta$, keeping the ratio to the other parameters z, T, γ fixed, and subtract it from F. This yields a regularized integrand

$$\bar{F}(z, \epsilon, \epsilon') = D_\gamma(\epsilon - \epsilon') \left\{ \frac{-\tanh(E/2T)}{4E} \sum_{\eta = \pm} \frac{A_\eta + \epsilon\epsilon'}{\epsilon'^2 - Q_\eta^2} + \frac{1}{2} \tanh\left(\frac{\epsilon}{2T} \right) \frac{\epsilon' - \epsilon}{(\epsilon' - \epsilon)^2 - z^2} \right\}. \tag{17}$$

We have checked that this difference is of order $\mathcal{O}[(|\epsilon| + |\epsilon'|)^{-2}]$ in the direction $\epsilon = \epsilon'$ and $\mathcal{O}[(|\epsilon| + |\epsilon'|)^{-5}]$ in all other directions of the $\epsilon\epsilon'$-plane. In addition, there are no poles for real-valued ϵ, ϵ' so that the integral of $\bar{F}(z, \epsilon, \epsilon')$ is uniformly convergent.

Comparison of equations (9) and (17) shows that this procedure is exactly equivalent to subtracting the spectral function of the normal conductor from the BCS expression $L(z, \epsilon, \epsilon')$. We hence find a form of the conductivity as anticipated in equation (8), where the BCS correction is

$$\begin{aligned} \delta\sigma_{\mathrm{BCS}}(z) &= \frac{\sigma_0 \gamma}{\mathrm{i}z} \int \mathrm{d}\epsilon \, \mathrm{d}\epsilon' [\bar{F}(z, \epsilon, \epsilon') + \bar{F}(z, \epsilon', \epsilon)] \\ &= 2 \frac{\sigma_0 \gamma}{\mathrm{i}z} \int \mathrm{d}\epsilon \, \mathrm{d}\epsilon' \bar{F}(z, \epsilon, \epsilon'). \end{aligned} \tag{18}$$

In the second line, we have exploited the absolute convergence of the integral of $\bar{F}(z, \epsilon, \epsilon')$, to interchange the order of the ϵ, ϵ' integrations for the second term.

The BCS correction $\delta\sigma_{\mathrm{BCS}}(z)$ (see equation (8)) is now evaluated as has been done by Mattis and Bardeen for real frequencies. We remark however that since the subtracted integrand *in toto* converges uniformly, no cutoff is required. The ϵ' integration of the quantity $\bar{F}(z, \epsilon, \epsilon')$ is easily performed with contour techniques. We first note that thanks to the subtraction, the integrand is at most of order $|\epsilon'|^{-2}$ as $|\epsilon'| \to \infty$ at fixed ϵ so that we can close the contour by a semi-circle at infinity in the upper half-plane. We then observe that the second term between the curly brackets in equation (17) is an odd function of $\epsilon' - \epsilon$ and vanishes upon integrating

over ϵ'. The first term has simple poles occurring at $\epsilon' = \epsilon + i\gamma$, and at $\epsilon' = Q_\pm$. The square root $Q_\pm$ must be taken with a cut along the real positive axis[4], in such a way that $\mathrm{Im}(Q_\pm) > 0$. Evaluation of the corresponding residues gives the result

$$\delta\sigma_{\mathrm{BCS}}(z) = i\frac{\sigma_0\gamma}{2z} \int_{-\infty}^{\infty} \frac{d\epsilon}{E} \tanh\left(\frac{E}{2T}\right) \sum_{\eta=\pm} G_\eta(z, \epsilon), \tag{19}$$

where

$$G_\pm(z, \epsilon) = \frac{i\gamma(A_\pm + \epsilon Q_\pm)}{Q_\pm[(Q_\pm - \epsilon)^2 + \gamma^2]} + \frac{A_\pm + \epsilon(\epsilon + i\gamma)}{(\epsilon + i\gamma)^2 - Q_\pm^2}. \tag{20}$$

We note that both terms in equation (20) have a pole at $\epsilon = Q_\pm - i\gamma$, but the residues compensate each other exactly. This creates numerical problems, that can be avoided by rearranging into

$$G_\pm(z, \epsilon) = \frac{\epsilon^2 Q_\pm(z, E) + (Q_\pm(z, E) + i\gamma)A_\pm(z, E)}{Q_\pm(z, E)[\epsilon^2 - (Q_\pm(z, E) + i\gamma)^2]}, \tag{21}$$

where we have restored the arguments of $Q_\pm$ and $A_\pm$ for clarity and $E = E(\epsilon)$ as in equation (3). Note that the integrand in equation (19) is actually even in ϵ.

Equations (19) and (21), together with equation (8), provide the desired analytic continuation of the BCS conductivity to complex frequencies $z \in \mathbb{C}^+$. In appendix B, we verify that our formula, when considered for real frequencies ω, correctly reproduces the expression for $\sigma(\omega)$ quoted in [7, 8].

Consider now the special case of a purely imaginary frequency $z = i\xi$, that one needs for thermal equilibrium quantities calculated in the Matsubara imaginary-time formalism. By using the relations $Q_-(i\xi, \epsilon) = -Q_+^*(i\xi, -\epsilon)$ and $A_-(i\xi, \epsilon) = A_+^*(i\xi, -\epsilon)$, it is easy to verify that the functions $G_\pm(i\xi, \epsilon)$ satisfy the relation $G_-(i\xi, \epsilon) = G_+^*(i\xi, -\epsilon)$. Then, from equations (19) and (20) we obtain

$$\delta\sigma_{\mathrm{BCS}}(i\xi) = \frac{\sigma_0\gamma}{\xi} \int_{-\infty}^{+\infty} \frac{d\epsilon}{E} \tanh(E/2T)\, \mathrm{Re}[G_+(i\xi, \epsilon)]. \tag{22}$$

In the normal conductor limit ($\Delta \to 0$), $G_+(i\xi, \epsilon)$ becomes purely imaginary so that equation (22) vanishes, leaving only the Drude term in equation (8).

From this expression we can obtain the 'extremely anomalous' or 'dirty' limit $\gamma \gg \Delta$ considered by Mattis and Bardeen, by expanding in powers of $1/\gamma$, and retaining only the lowest order:

$$G_+(i\xi, \epsilon) \approx \frac{i}{\gamma} \frac{A_+(i\xi, \epsilon)}{Q_+(i\xi, \epsilon)}. \tag{23}$$

In this limit, the BCS correction $\delta\sigma_{\mathrm{BCS}}$ (equation (22)) no longer depends on γ. Since the approximation is valid for small frequencies $\xi \ll \gamma$, the Drude contribution in (8) becomes $\sigma_{\mathrm{Dr}}(i\xi) \approx \sigma_0$ in this limit.

3.3. Numerical efficiency

Kramers–Kronig relations are commonly used to calculate conductivities or dielectric functions at imaginary frequencies starting from the known representations at real frequencies (see, e.g., [11]). This is particularly useful if one wants to extrapolate experimental data to the imaginary axis. The formulation we use here,

$$\delta\sigma_{\mathrm{BCS}}(i\xi) = \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega\, \mathrm{Im}\sigma(\omega)}{\omega^2 + \xi^2}, \tag{24}$$

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[4] In terms of a square root `sqrt` whose cut is along the negative real axis, $Q_\pm = i\,\mathtt{sqrt}[\Delta^2 - (E \pm z)^2]$.
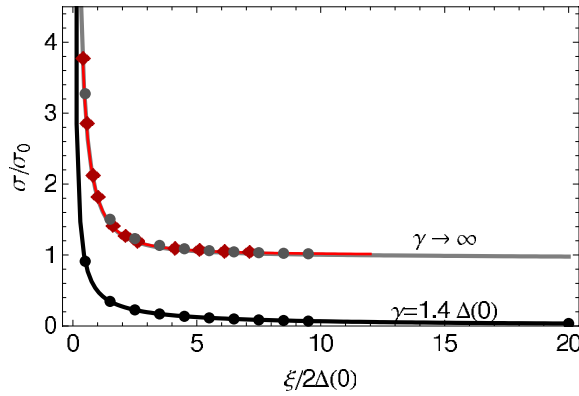
Figure 1. BCS conductivity evaluated along the imaginary positive frequency axis. Lines: direct analytical continuation derived in equations (8) and (21), (22) in the clean ($\gamma \approx 1.4\,\Delta$, black line) and dirty ($\gamma \approx 1400\,\Delta$, light gray line) case and the extremely anomalous limit ($\gamma \gg \Delta$, dark gray line) using equation (23). Dots: Kramers–Kronig continuation from the real frequency conductivities by Zimmermann *et al* [7], Berlinsky *et al* [8], following the scheme summarized in equation (24) for the clean and dirty case. Diamonds: Kramers–Kronig continuation in the extremely anomalous case starting from the complex conductivity $\sigma(\omega)$ of [5].

involves the imaginary part of the conductivity (an odd function in ω). We thus avoid the $\delta(\omega)$ singularity [5, 8] that occurs in $\mathrm{Re}\,\sigma(\omega)$. Equation (24) is numerically quite inconvenient in the BCS theory since it requires two integrations (over ϵ and ω). The calculations are more easy for the direct analytical continuation, equations (8), (21), (22), where a single integral must be done that in addition converges rapidly. The agreement between the two representations is excellent as one can see from figure 1: the solid lines are obtained with our analytical continuation while the dots result from the Kramers–Kronig scheme. Our results also reproduce the dirty limit based on equation (23) ($\gamma \gg \Delta$) considered by Mattis and Bardeen [5] for real frequencies. All the numerical work has been performed using the approximate form of the BCS gap function $\Delta(T)$ given in [13, 14].

4. Applications

4.1. Supercurrent response

As a first example, we show that the BCS response function can be analyzed in a quite simple way at imaginary frequencies. We recall the discussion of Berlinsky *et al* [8] who attribute the weight of the peak $\propto \delta(\omega)$ in $\mathrm{Re}\,\sigma(\omega)$ to the Cooper pair condensate and the response at nonzero frequencies to thermally excited quasiparticles ($\omega \ll 2\Delta$) and to 'direct' excitations across the gap ($\omega \geqslant 2\Delta$). All three contributions together fulfill the sum rule for the oscillator strength in the conductivity. The first and second contributions are associated with the superfluid and the normal fluid of the two-fluid model due to Gorter and Casimir [4, 15], whose weights vary with T in a way similar to the Gorter–Casimir rule. The third contribution is found to be approximately constant over a large range of T.

We have tried to evaluate the imaginary frequency representation of these three contributions, taken separately. We recognized, however, that the 'thermal' and 'direct' parts of the quasiparticle response (i.e. the first and second terms in equation (4)) show a

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branch cut in the ϵ'-plane so that only their sum can be combined in the same way as in equations (13)–(16). In what follows, we hence consider a slightly different 'two-fluid split' of the conductivity that is closer to the Gorter and Casimir approach: a supercurrent part that does not show any relaxation (or scattering) and a resistive current whose frequency response is Drude-like. In addition, we reproduce the *T*-dependent weight of a logarithmically divergent contribution to the conductivity at low frequencies given in [8]. This behavior which is usually attributed to the divergent mode density of the BCS approach at the gap that is not smoothed out in the Chambers-like way disorder is introduced by Mattis and Bardeen [5].

Along the imaginary frequency axis, the superfluid response corresponds to the weight of the pole at $\xi = 0$. In terms of an effective superfluid plasma frequency,

$$\xi \to 0: \qquad \sigma(\mathrm{i}\xi) \approx \frac{\varepsilon_0 \omega_\mathrm{s}^2(T)}{\xi} + B(T)\log(\Delta/\xi) + C(T), \tag{25}$$

where the logarithmic term is discussed below.

Comparing to equation (8), we get ω_s by considering $\delta\sigma_\mathrm{BCS}(\mathrm{i}\xi)$ in the limit $\xi \to 0$. Our starting point is then equation (22). For $\xi \to 0$, the integrand $G_+(\mathrm{i}\xi, \epsilon)$ (equation (21)) as a function of ϵ exhibits two relevant features: first, a 'narrow peak' originating from the term $1/Q_+(\mathrm{i}\xi, E(\epsilon))$, its width in ϵ scales like $(\xi\Delta)^{1/2}$. Setting $\epsilon = x(\xi\Delta)^{1/2}$, we get

$$\omega_\mathrm{s,1}^2 = \frac{2\sigma_0\Delta}{\varepsilon_0}\int_0^{+\infty}\mathrm{d}x\,\tanh\frac{E(\epsilon)}{2T}\,\mathrm{Re}\left[\frac{2\mathrm{i}}{\sqrt{x^2+2\mathrm{i}}}\right] \xrightarrow{\xi\to 0} \pi\frac{\sigma_0\Delta}{\varepsilon_0}\tanh(\Delta/2T), \tag{26}$$

where convergence at the upper limit is secured by taking the real part. In the last step, we have taken the lowest order in the expansion of $E(\epsilon) = \Delta\sqrt{1+(\xi/\Delta)x^2}$ for small ξ. In the limit $T \to 0$, we recognize here the integral term in equation (14) of Berlinsky *et al* [8].

The second feature in $G_+(\mathrm{i}\xi, \epsilon)$ is a 'broad background' whose width as a function of ϵ is set by Δ and γ. This gives a negative contribution to ω_s^2:

$$\omega_\mathrm{s,2}^2 = -4\frac{\sigma_0\gamma}{\varepsilon_0}\Delta^2\int_0^\infty\mathrm{d}\epsilon\,\frac{\tanh(E/2T)}{\sqrt{\Delta^2+\epsilon^2}(\gamma^2+4\epsilon^2)}$$

$$\xrightarrow{T\to 0} -4\frac{\sigma_0}{\varepsilon_0}\Delta(0)\frac{\mathrm{arcsec}(2\Delta(0)/\gamma)}{\sqrt{4-(\gamma/\Delta(0))^2}}. \tag{27}$$

The zero-temperature limit given in equation (15) of Berlinsky *et al* [8] is identical to the sum of equations (26,27). Recall that in this paper, Δ always denotes the temperature-dependent gap.

Figure 2 shows the behavior of $\omega_s^2 = \omega_\mathrm{s,1}^2 + \omega_\mathrm{s,2}^2$ as a function of temperature and scattering rate. We normalize to the plasma frequency given by the total carrier density, $\omega_\mathrm{pl}^2 = \sigma_0\gamma/\varepsilon_0$. This suggests an interpretation in terms of a superfluid order parameter [15, 16]. Indeed, the superfluid response identically vanishes for temperature higher than the critical temperature and goes to a constant in the limit $T \ll T_c$. Note that at low temperatures, ω_s still depends on the disorder in the sample via the parameter γ (see also in the following).

4.2. Logarithmic correction

It is well known that at nonzero temperature, the BCS conductivity shows a logarithmically divergent term at low frequencies. The weight $B(T)$ of this term (see equation (25)) can also be calculated along imaginary frequencies, by improving on the small-ξ limit discussed before. The 'narrow peak' of equation (26) allows us to take into account the small ϵ expansion of the
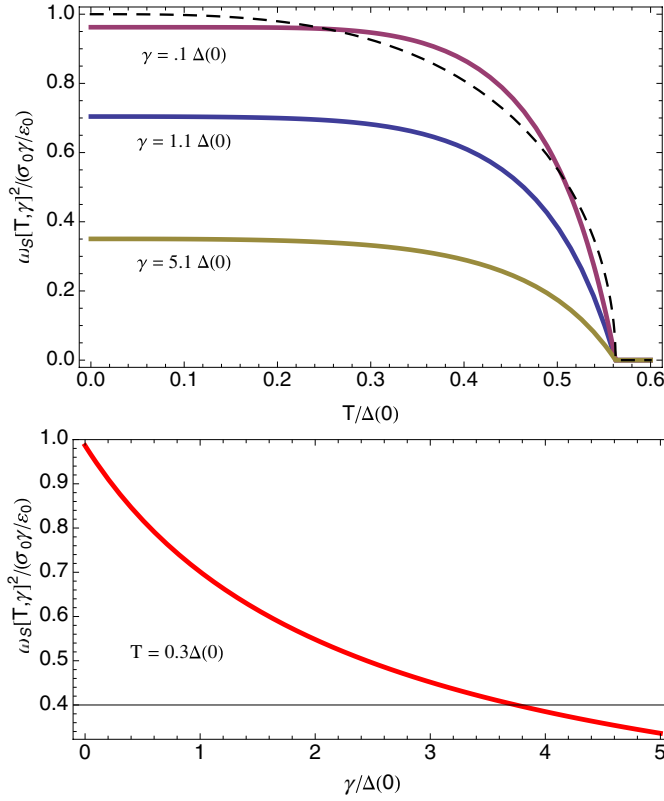
Figure 2. Superconducting order parameter for different disorder strengths. We plot the superfluid part of the current response function, expressed in terms of the superfluid plasma frequency squared, ω_s^2, and normalized to the weight of the high-frequency (above gap) response $\omega_{pl}^2 = \sigma_0\gamma/\varepsilon_0$. This is plotted as a function of the temperature (top) and of the scattering rate γ (bottom). Dashed line: normalized BCS gap $\Delta(T)/\Delta(0)$. The superconducting fraction vanishes above the critical temperature $T_c \approx 0.56\,\Delta(0)$ and becomes almost constant for $T \ll T_c$. In this limit, ω_s^2 still depends on γ and reaches the normal-phase plasma frequency ω_{pl}^2 only in the pure limit $\gamma \to 0$.

temperature-dependent factor $\tanh E/2T$. The $\mathcal{O}(\epsilon^2)$ term of this expansion gives then the following integral in the scaled energy $x = \epsilon/(\xi\Delta)^{1/2}$:

$$\sigma(\mathrm{i}\xi) - \frac{\varepsilon_0\omega_s^2(T)}{\xi} \approx \frac{\sigma_0\Delta}{T}\mathrm{sech}^2(\Delta/2T)\int_0^{x_c}\mathrm{d}x\,\mathrm{Re}\left[\frac{\mathrm{i}x^2}{\sqrt{x^2+2\mathrm{i}}}\right]. \tag{28}$$

A cutoff x_c is needed for the integral in order to reproduce the convergent behavior of the original expression. Inspection of the full integrand shows that it starts to decrease at $\epsilon \geqslant \Delta$ so that we take $x_c = (\Delta/\xi)^{1/2}$. The integral can then be evaluated explicitly and gives to leading order at small ξ, $\log x_c = \frac{1}{2}\log(\Delta/\xi)$. We thus read off the coefficient $B(T)$ in equation (25):

$$B(T) = \frac{\sigma_0\Delta}{2T}\mathrm{sech}^2(\Delta/2T). \tag{29}$$

It is easy to check that in the regime $\omega \ll T \ll \Delta$, our result (analytically continued to real frequencies) overlaps with the formulas (20) and (21) of Berlinsky *et al* [8]. Their result apparently allows for larger frequencies up to the order T, but is restricted to $\omega, T \ll \Delta$.
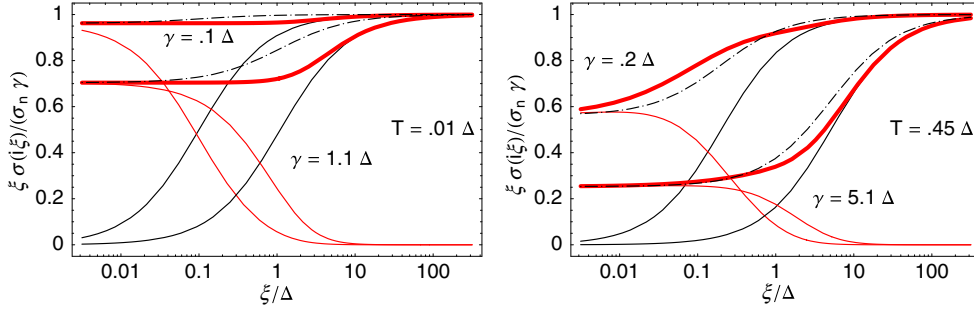
Figure 3. Effective oscillator strength in the frequency range $0 \leqslant \omega \leqslant \xi$, as measured by the ratio $\xi \sigma(\mathrm{i}\xi)/(\sigma_0 \gamma)$. The thick curve is based on the full BCS conductivity, the (lower) thin curve on the Drude part only, $\xi/(\gamma + \xi)$. The (upper) thin curve shows the BCS correction. Dash-dotted line: the two-fluid-like model of equation (30) with the 'order parameter' $\eta(T)$ matched to reproduce the small-ξ limit. We consider a 'clean' and a 'dirty' case at a low temperature (top) and at a higher temperature (bottom). Note that Δ is always the T-dependent gap here. The shoulder at low frequencies in the 'clean and warm' case (right, top curve, $\gamma = 0.2\,\Delta$, $T = 0.45\,\Delta$) is probably related to the logarithmic contribution in equation (25).

4.3. Partial sum rule

We now make a connection between the conductivity at imaginary frequencies and an effective oscillator strength, integrated over in a finite range of frequencies. More precisely, we consider the product $\xi \sigma(\mathrm{i}\xi)$ that is plotted in figure 3. A Kramers–Kronig relation, similar to equation (24), but based on $\mathrm{Re}\,\sigma(\omega)$, shows that $\xi \sigma(\mathrm{i}\xi)$ is an non-decreasing function of ξ that collects the oscillator strength in the frequency range $0 \leqslant \omega \leqslant \xi$. The sum rule for $\mathrm{Re}\,\sigma(\omega)$ thus implies that $\xi \sigma(\mathrm{i}\xi) \to \gamma \sigma_0$ for $\xi \to \infty$. According to equation (25), the limiting value of $\xi \sigma(\mathrm{i}\xi)$ for $\xi \to 0$ is $\varepsilon_0 \omega_s^2(T) =: \eta(T)\gamma\sigma_0$ where $0 \leqslant \eta(T) \leqslant 1$ is similar to a superconducting order parameter. A normal conductor necessarily has $\eta(T) = 0$ because its dc conductivity is finite.

We compare the product $\xi \sigma(\mathrm{i}\xi)$ in figure 3 to an effective two-fluid model with the conductivity (dash-dotted lines):

$$\sigma_{2\mathrm{F}}(\mathrm{i}\xi) = \eta(T)\frac{\sigma_0\gamma}{\xi} + (1 - \eta(T))\frac{\sigma_0\gamma}{\xi + \gamma}, \tag{30}$$

where the damping rate for the normal fluid contribution (second term) is set to γ. The overall agreement is quite good, but the logarithmic correction is of course not captured by this approach. We also note that the 'order parameter' $\eta(T)$ introduced in this way does not coincide with the gap function $\Delta = \Delta(T)$, see equations (26, 27). In particular, it depends both on temperature and the dissipation rate and does not reach $\eta(T \to 0) = 1$ as long as γ is nonzero (see also figure 2).

4.4. Plasmon self-energy

As a further example, we consider the electromagnetic self-energy of a bulk superconductor at zero temperature. This quantity can be calculated from the oscillator strength of the bulk plasmon oscillation [17]. Recall that plasmons are collective, longitudinal oscillations of the carrier density that are up-shifted in energy through the Coulomb interaction [18, 19]. We

calculate here the self-energy in the local limit and find the longitudinal (density) response from the dielectric function:

$$\varepsilon(\omega) = 1 + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}. \tag{31}$$

The bulk plasmon dispersion relation is indeed given by $\varepsilon(\omega) = 0$ with solutions in the lower half-plane $\mathbb{C}^-$. Along the real frequency axis, the spectral density broadens and shifts, and applying the logarithmic argument theorem, the self-energy is given by [17, 20–22]

$$E = \int_0^\infty \frac{d\omega}{\pi} \frac{\omega}{2} \partial_\omega \text{Im} \log\left(\frac{1}{\varepsilon(\omega)}\right). \tag{32}$$

Performing a partial integration and shifting the integration path to imaginary frequencies, we get

$$E = \text{Im} \int_0^\infty \frac{d\omega}{2\pi} \log \varepsilon(\omega) = \int_0^\infty \frac{d\xi}{2\pi} \log \varepsilon(i\xi). \tag{33}$$

where the $\log \xi$ divergence at $\xi \to 0$ is integrable. For a normal conductor ($\varepsilon(\omega)$ in Drude form), this is easily evaluated in terms of two characteristic frequencies, Ω_{pl} and γ:

$$E_D = \text{Re}\left[\frac{\Omega_{pl}}{2} - i\frac{\Omega_{pl}}{\pi} \log \frac{\Omega_{pl}}{\gamma}\right], \qquad \Omega_{pl} = \sqrt{\omega_{pl}^2 - \gamma^2/4} - i\gamma/2, \tag{34a}$$

where $\omega_{pl} = \sqrt{\gamma \sigma_0/\varepsilon_0}$ is again the plasma frequency. The case $\gamma \ll \omega_{pl}$ is the relevant one for most materials. In the limit $\gamma \to 0$, we recover the zero-point energy $\omega_{pl}/2$ for an undamped plasmon (dielectric function $\varepsilon(\omega) = 1 - \omega_{pl}^2/\omega^2$). At nonzero γ, corrections are due to the renormalization of the real part of Ω_{pl} and the logarithmic term. The latter can be interpreted as the contribution of zero-point fluctuations of the path that is responsible for the damping of the plasmon mode [23, 24]. Alternatively, it represents how the modes of the path shift in frequency, due to the coupling to the plasma oscillator [17]. We quote here only for completeness the overdamped limit ($\omega_{pl} < \gamma/2$):

$$E_D = -\sum_\pm \frac{\xi_\pm}{2\pi} \log \frac{\xi_\pm}{\gamma}, \qquad \xi_\pm = \gamma/2 \pm \sqrt{\gamma^2/4 - \omega_{pl}^2}, \tag{34b}$$

although the Mattis–Bardeen theory for an impure superconductor [5] is clearly no longer valid in this limit.

For the superconductor, the self-energy $E = E_S$ follows very closely the normal conductor, as can be seen from figure 4. The difference arises from collective modes within the gap and modifies E_S by an amount of order Δ. The plot features the transition to the dirty limit [5] at $\gamma \sim \Delta(0)$ and a saturation of $E_S - E_D$ in the overdamped limit.

4.5. Casimir free energy of a superconducting cavity

Another example where the knowledge of optical response functions at imaginary frequencies is not only highly useful, but virtually indispensable, are electromagnetic self-energies like the Casimir interaction. Indeed, the calculation of these energies is practically impossible to perform along real frequencies because of a highly oscillatory integrand. Along the imaginary frequency axis the integrand becomes smooth and rapidly convergent, which is similar to what happens in equation (33). But the Casimir effect in a superconducting cavity is very promising also from a physical viewpoint because the superconductor offers the possibility of controlling dissipation. The influence of the latter on the Casimir free energy remains indeed
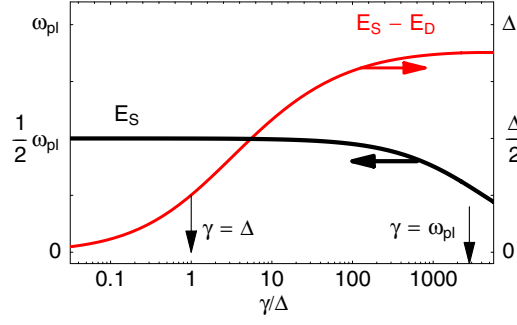
Figure 4. Electromagnetic self-energy E_S (equation (33) with the BCS dielectric function) of the bulk plasmon mode in a superconductor versus the impurity parameter γ (logarithmic scale), at zero temperature. Thick black line: self-energy E_S, in units of the plasma frequency $\omega_{pl} \approx 2800\,\Delta(0)$ (left scale). Thin line: difference $E_S - E_D$ to the normal conductor, equation (34*a*), in units of the (zero temperature) BCS gap $\Delta = \Delta(0)$ (right scale). We take for E_D a Drude dielectric function with the same plasma frequency ω_{pl} and damping rate γ.

an open question [10, 25, 26]: two conflicting viewpoints have been raised as to whether the dc conductivity of the mirrors should be included in the modeling or not. It has therefore been proposed to resolve this issue by considering the Casimir energy of a superconducting cavity, more precisely its change across the critical temperature [3, 11, 27]. Earlier calculations in the framework of the BCS theory [11] had to use a Kramers–Kronig analytic continuation of the optical response functions to imaginary frequencies (equation (24)), leading to a significant computational overhead. The scheme developed in this work requires less integrations and enables us to perform numerical analysis in a much more flexible and precise way.

We recall that the Casimir free energy per unit area [9] between two plates (separation L, temperature T) is given by a sum over the Matsubara frequencies $\xi_n = 2\pi nT$:

$$\mathcal{F}(L, T) = \frac{T}{2\pi} \sum_{n=0}^{\infty}{}' \int_0^\infty k \, \mathrm{d}k \sum_p \log D_p(\mathrm{i}\xi_n, k)$$

$$D_p(\omega, k) = 1 - r_p^2(\omega, k) \exp(-2\kappa L),$$

(35)

where the primed sum weighs the zeroth term by $1/2$ and the last sum is over polarizations $p \in \{\text{TE}, \text{TM}\}$. The simplest model for the reflection amplitudes is the Fresnel equations :

$$r_{\text{TE}}(\omega, k) = \frac{\kappa - \kappa_m}{\kappa + \kappa_m}, \qquad r_{\text{TM}}(\omega, k) = \frac{\epsilon(\omega)\kappa - \kappa_m}{\epsilon(\omega)\kappa + \kappa_m}$$

(36)

with the notation $\kappa = \sqrt{k^2 - \frac{\omega^2}{c^2}}$ and $\kappa_m = \sqrt{k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}}$ (for $\omega \in \mathbb{C}^+$, $\operatorname{Re}\kappa \geqslant 0$ and $\operatorname{Im}\kappa \leqslant 0$). In the numerical calculation of the Matsubara sum from equation (35), the sum has been cut off at a sufficiently high [28] value of n and the zeroth summand ($\xi_0 = 0$) has been evaluated using the results of section 4.1.

Figure 5 shows the temperature dependence of the Casimir free energy calculated for different theoretical descriptions using the parameters for niobium. Both the BCS description and the simple two-fluid model (equation (30) with $\eta(T) = 1 - (T/T_c)^4$) predict a sudden change of the Casimir energy across the superconducting transition. As estimated in [3, 11, 27], this effect should be observable in experiments.

Having in mind the discussion from section 4.1, it is not surprising that there is an offset between the BCS curve and the two-fluid model (or plasma) curve at low temperatures. This
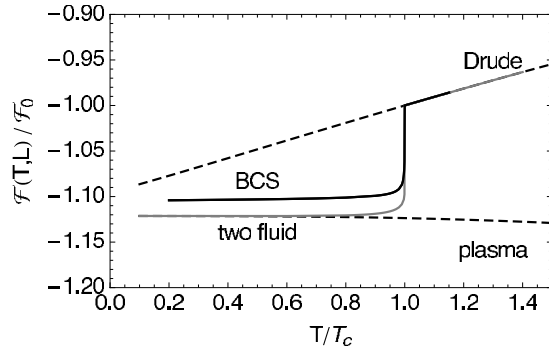
Figure 5. Casimir free energy (per unit area) versus temperature in a planar cavity of length $L = 100\,\lambda_p$ where $\lambda_p = 2\pi c/\omega_{pl}$ is the plasma penetration depth. For numerical work we used values for niobium [29] $\omega_{pl} = 1.408 \times 10^{16}$rad s^{-1}, $\gamma = 2.44 \times 10^{14}$rad s^{-1}, $T_c = 9.25$ K, and $\Delta(0) = 1.7\,T_c \approx 8.5 \times 10^{-4}\gamma$ (dirty limit). Energies are normalized to $\mathcal{F}_0 \approx 0.2\,T_c/L^2$, the absolute value of the free energy in the Drude model at T_c. The Matsubara sum (35) is computed including up to 10 000 terms at the lowest temperatures.

is because impurity scattering renormalizes the supercurrent plasma frequency ω_s, leading to a smaller effective oscillator strength in the BCS conductivity, in particular for low ξ.

5. Conclusions

In this paper we have analyzed the analytical continuation of the bulk BCS conductivity to imaginary frequencies. The final result is numerically more efficient than previously used schemes based on Kramers–Kronig relations. Our approach simplifies the direct calculation of most of the involved integrals, and displays clearly convergence issues. It also illustrates how the conductivity of the *normal* component emerges naturally from the BCS framework. The superconducting correction vanishes as the BCS gap $\Delta \to 0$ and is given in terms of a rapidly convergent integral. The result is convenient for both analytical and numerical computations of the optical response of superconductors. We have illustrated this by rederiving in a simple way the weight of the supercurrent response, including its dependence on temperature T and the mean free path $\ell = v_F/\gamma$, and the weight of the logarithmically divergent term in the conductivity. As another illustration, we have given numerical calculations of the change in the bulk plasmon self-energy at zero temperature, showing how impurity scattering increases the self-energy, although the superconducting gap lies well below the bulk plasmon resonance. Finally we have demonstrated how the Casimir energy for the case of a cavity made of superconducting material can be calculated numerically in an efficient way and recovered the energy change in the superconducting transition.

A more detailed investigation including the discussion of the distance dependence of the free energy and possibilities to tune Casimir energies through the dissipation rate γ will be presented elsewhere. Future work will also discuss the temperature-dependence of the self-energy. Preliminary studies have revealed features (related to the behavior of $\sigma(i\xi)$ near $\xi = 0$) that lead to similar physical effects as in the thermal correction to the Casimir energy between metals or ideal conductors [10, 25, 26]. It would also be interesting to extend the calculations beyond the local (or macroscopic) limit, taking into account a finite wave-vector **q**, as done by Pöpel [30]. This would also provide a natural scheme for calculating the electromagnetic self-energy of a superconducting surface [31, 32].

Appendix A. Alternative calculation for the normal metal

This approach is closer to the way in which Mattis and Bardeen performed the integration. We start by combining equations (6), (9) and get for a complex frequency $z \in C^+$:

$$\tilde{\sigma}(z) = \frac{\sigma_0 \gamma}{2 i z} \int d\epsilon \, d\epsilon' [\tanh(\epsilon'/2T) - \tanh(\epsilon/2T)] \frac{\epsilon' - \epsilon}{(\epsilon' - \epsilon)^2 - z^2} D_\gamma(\epsilon - \epsilon') \, e^{i(\epsilon + \epsilon')/2\Lambda},$$

(A.1)

where the limit $\xi \to 0$ of the integral has still to be subtracted. We have introduced a convergence factor $e^{i(\epsilon + \epsilon')/2\Lambda}$ which ensures absolute convergence also in the sum variable $\epsilon + \epsilon'$, when analytically continued into the upper half-plane. The order of integrations is now immaterial. Following Mattis and Bardeen, we can handle separately the two terms with the hyperbolic functions: integrate the term proportional to $\tanh(\epsilon'/2T)$ over ϵ (closing the contour in the upper half of the complex ϵ-plane) and over ϵ' the term involving $\tanh(\epsilon/2T)$. Summing the two results, this gives

$$\tilde{\sigma}(z) = \frac{\sigma_0 \gamma}{\pi i z} \int d\epsilon \tanh(\epsilon/2T) \frac{\pi i \gamma \, e^{i\epsilon/\Lambda}}{z^2 + \gamma^2} (e^{-\gamma/2\Lambda} - e^{iz/2\Lambda}).$$

(A.2)

Due to the convergence factor, the integral over ϵ does not vanish by parity. It can be calculated by considering a rectangular contour in the complex ϵ-plane that encloses a strip of height $2\pi T$ in the upper half-plane. In this way, we find

$$\tilde{\sigma}(z) = \frac{\sigma_0 \gamma}{\pi i z} \frac{2\pi i T}{\sinh(\pi T/\Lambda)} \frac{\pi i \gamma}{z^2 + \gamma^2} (e^{-\gamma/2\Lambda} - e^{iz/2\Lambda}) \to \frac{\sigma_0 \gamma^2}{i z(\gamma - i z)},$$

(A.3)

where the limit $\Lambda \to \infty$ was taken in the last step. We still have to subtract the limiting case $z \to 0$, i.e. the diamagnetic term. This gives the Drude conductivity $\sigma_{Dr}(z)$ of equation (12).

Appendix B. Comparison with Zimmermann's formula

In this Appendix, we prove that our formula for the conductivity, equations (8), (19) and (20), when considered for real (positive) frequencies ω is equivalent to the following expression, quoted in [7], and reproduced here for the convenience of the reader (we take $\hbar = 1$):

$$\sigma(\omega) = i \frac{\sigma_0 \gamma}{2\omega} \left(J + \int_\Delta^\infty dE \, I_2 \right),$$

(B.1)

$$J(\omega \leqslant 2\Delta) = \int_\Delta^{\omega + \Delta} dE \, I_1,$$

(B.2)

$$J(\omega \geqslant 2\Delta) = \int_\Delta^{\omega - \Delta} dE \, I_3 + \int_{\omega - \Delta}^{\omega + \Delta} dE \, I_1,$$

(B.3)

$$I_1 = \tanh\left(\frac{E}{2T}\right) \left\{ \left[1 - \frac{A_-}{P_4 P_2}\right] \frac{1}{P_4 + P_2 + i\gamma} - \left[1 + \frac{A_-}{P_4 P_2}\right] \frac{1}{P_4 - P_2 + i\gamma} \right\},$$

(B.4)

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$$I_2 = \tanh\left(\frac{E+\omega}{2T}\right)\left\{\left[1 + \frac{A_+}{P_1 P_2}\right]\frac{1}{P_1 - P_2 + \mathrm{i}\gamma} - \left[1 - \frac{A_+}{P_1 P_2}\right]\frac{1}{-P_1 - P_2 + \mathrm{i}\gamma}\right\}$$
$$+ \tanh\left(\frac{E}{2T}\right)\left\{\left[1 - \frac{A_+}{P_1 P_2}\right]\frac{1}{P_1 + P_2 + \mathrm{i}\gamma} - \left[1 + \frac{A_+}{P_1 P_2}\right]\frac{1}{P_1 - P_2 + \mathrm{i}\gamma}\right\},$$

(B.5)

$$I_3 = \tanh\left(\frac{E}{2T}\right)\left\{\left[1 - \frac{A_-}{P_3 P_2}\right]\frac{1}{P_3 + P_2 + \mathrm{i}\gamma} - \left[1 + \frac{A_-}{P_3 P_2}\right]\frac{1}{P_3 - P_2 + \mathrm{i}\gamma}\right\},$$

(B.6)

where

$$P_1 = \sqrt{(E+\omega)^2 - \Delta^2}, \qquad P_2 = \sqrt{E^2 - \Delta^2} \qquad\qquad (B.7)$$

$$P_3 = \sqrt{(E-\omega)^2 - \Delta^2}, \qquad P_4 = \mathrm{i}\sqrt{\Delta^2 - (E-\omega)^2}. \qquad (B.8)$$

In the relevant ranges of ω, the arguments of the square roots are non-negative real numbers, and the square roots are then defined to be positive. This conductivity differs from our equations (8), (19) and (21) by the presence of the frequency ω in some of the integration boundaries. To demonstrate the equivalence with our formulas, we show as a first step that equations (B.1)–(B.8) can be split into a Drude term and a BCS correction, as in equation (8). We note that the quantities P_1, P_3 and P_4 above are related to our quantities $Q_\pm(\omega, E)$ as follows:

$$P_1(\omega, E) = Q_+(\omega + \mathrm{i}\,0^+, E), \qquad\qquad (B.9)$$

$$P_3(\omega \geqslant 2\Delta, E) = Q_-(\omega + \mathrm{i}\,0^+, E) \qquad \text{for} \quad \Delta \leqslant E \leqslant \omega - \Delta \qquad (B.10)$$

and

$$P_4(\omega \leqslant 2\Delta, E) = Q_-(\omega + \mathrm{i}\,0^+, E) \qquad \text{for} \quad \Delta \leqslant E \leqslant \omega - \Delta. \qquad (B.11)$$

$$P_4(\omega \geqslant 2\Delta, E) = Q_-(\omega + \mathrm{i}\,0^+, E) \qquad \text{for} \quad \omega - \Delta \leqslant E \leqslant \omega + \Delta. \qquad (B.12)$$

Using relations (B.9)–(B.12) above, we verify that for all non-negative real frequencies ω, equations (B.1)–(B.8) can be recast in the following compact form

$$\sigma(\omega) = \frac{\mathrm{i}\sigma_0\gamma}{2\omega}H(\omega) + \delta\sigma(\omega), \qquad\qquad (B.13)$$

where

$$H(\omega) = \int_\Delta^\infty \mathrm{d}E\left[\tanh\left(\frac{E+\omega}{2T}\right)B_-(\omega, E+\omega) - \theta(E - \Delta - \omega)\tanh\left(\frac{E}{2T}\right)B_-(\omega, E)\right]$$

(B.14)

and

$$\delta\sigma(\omega) = \int_\Delta^\infty \mathrm{d}E\,\tanh\left(\frac{E}{2T}\right)\sum_{\alpha=\pm}B_\alpha(\omega, E), \qquad\qquad (B.15)$$

with

$$B_\pm(\omega, E) = \left(1 - \frac{A_\pm}{Q_\pm P_2}\right)\frac{1}{Q_\pm + P_2 + \mathrm{i}\gamma} - \left(1 + \frac{A_\pm}{Q_\pm P_2}\right)\frac{1}{Q_\pm - P_2 + \mathrm{i}\gamma}. \qquad (B.16)$$

We can now show that the first term on the rhs of equation (B.13) coincides with the normal metal contribution. For this purpose, consider the quantity $H(\omega)$. If the integral of $\tanh(E/2T)B_-(\omega, E)$ were absolutely convergent at infinity, it would be possible to perform

the shift $E + \omega \to E$ in the integration variable E in the first term between the square brackets of equation (B.14), and then we would find that $H(\omega)$ is zero. However, since for large values of E, the quantity $\tanh(E/2T)B_-(\omega, E)$ has the asymptotic expansion

$$\tanh(E/2T) \, B_-(\omega, E) = \frac{2}{\omega + i\gamma} + \mathcal{O}(E^{-2}), \tag{B.17}$$

its integral diverges at infinity. Therefore, the shift of integration variable is not permitted, and the conclusion that $H(\omega)$ vanishes is not warranted. To evaluate $H(\omega)$, consider the following cut-off function $f_\Lambda(E)$:

$$f_\Lambda(E) = \frac{\Lambda^2}{E^2 + \Lambda^2}. \tag{B.18}$$

Thanks to the difference appearing in the square brackets of equation (B.14), the integral over E is absolutely convergent at infinity, we have the identity

$$H(\omega) = \lim_{\Lambda \to \infty} \int_\Delta^\infty dE \left[\tanh\left(\frac{E + \omega}{2T}\right) B_-(\omega, E + \omega) \right.$$
$$\left. - \theta(E - \Delta - \omega) \tanh\left(\frac{E}{2T}\right) B_-(\omega, E) \right] f_\Lambda(E). \tag{B.19}$$

Since, for any $\Lambda < \infty$, the cut-off integrals of the two terms between the square brackets are individually convergent at infinity, it is now legitimate to shift the integration variable in the first term. After we do that, we get

$$H(\omega) = \lim_{\Lambda \to \infty} \int_{\Delta + \omega}^\infty dE \, \tanh\left(\frac{E}{2T}\right) B_-(\omega, E)[f_\Lambda(E - \omega) - f_\Lambda(E)]. \tag{B.20}$$

For large values of Λ the difference between the cut-off functions is appreciably different from zero only for large values of E, and therefore, we can replace $\tanh(E/2T)B_-(\omega, E)$ by its large-E expansion, equation (B.17). Inserting into equation (B.20), the evaluation of the elementary E-integral results into

$$H(\omega) = \frac{2\omega}{\omega + i\gamma}. \tag{B.21}$$

After substituting this expression for $H(\omega)$ into equation (B.13), we indeed find as promised that its first term reproduces the normal-metal conductivity $\sigma_0 \gamma/(\gamma - i\omega)$. This shows that equations (B.1)–(B.8) for the BCS conductivity $\sigma(\omega)$, once recast in the form of equation (B.13), are indeed analogous to our split form equation (8) of the conductivity.

What then remains to show is that our expression for $\delta\sigma_{BCS}(z)$, given in equations (19), (21), when considered for real frequencies, coincides with the quantity $\delta\sigma(\omega)$ in equations (B.15), (B.16). This can be worked out in a straightforward way by making the change of variables $\epsilon \to \text{sgn}(\epsilon)E$ in equation (19). We omit details of this lengthy calculation for brevity.

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